Higher Order Hadronic Energy Level Shifts in Protonium from Effective Field Theory

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Abstract: Using effective field theory for a proton and antiproton bound in a Coulomb potential, the shift of the ground state energy level is calculated to arbitrary order in the scattering length. Including the next order contact interaction, the correction due to the effective range parameter can also be obtained.

It was first realized by Caswell and Lepage that the interactions between photons and electrons at energies well below the electron mass can be formulated in a purely non-relativistic way in a theoretical framework they called NRQED[1]. In contrast to QED which is a renormalizable theory, NRQED is non-renormalizable. But used as an effective field theory[2] only for phenomena at low energies, it can be used to derive higher order radiative corrections in a systematic way without invoking covariant theory like the Dirac equation and the corresponding Bethe-Salpeter equation for bound states, when the physics is inherently not relativistic[3]. Since its formulation, NRQED has now been used by Kinoshita and Nio[4] to calculate higher order corrections to hyperfine splitting in muonium. Hoang, Labelle and Zebarjad have also recently completed the order α^6 correction of the hyperfine splitting of the ground state in positronium[5] within the same framework.

The effective Lagrangian is a systematic expansion in p/M where p is a characteristic momentum and M is the heavy mass in the problem, i.e. the energy above which all degrees of freedom are integrated out. The theory is therefore only valid at momenta below M. In an ordinary atom this mass is given by the electron mass, while in a hadronic atom it would be the pion mass. Remember that the typical momentum in an atom is αm where α is the fine-structure constant. Each term in the expansion must obey the symmetries we want the system to have. The expansion coefficients can be found in two ways. In an ordinary atom where we know the underlying, fundamental theory which is QED, we can use perturbation theory at low energies and match scattering amplitudes

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calculated in NRQED to those calculated in full QED[3]. Similarly, in a hadronic atom one can estimate the expansion coefficients by matching scattering amplitudes in the effective theory to what one would obtain from a more fundamental theory based on exchange of pseudoscalar and vector mesons. QCD is strongly coupled at low energies and can thus not be used. Instead of matching to an underlying theory, one can also determine the effective couplings by directly comparing the obtained scattering amplitudes with experimental data. At very low energies these are parameterized in terms of scattering lengths and various effective ranges.

An effective field theory for non-relativistic systems of nucleons was first constructed by Weinberg[6]. It includes systematic counting rules so that one knows which diagrams to keep and which to discard at any order in p/M. Since then, this approach has been taken up by many others[7]. In particular, Kaplan, Savage and Wise[8] have studied in great detail the nucleon-nucleon system in the $^1S_0$ state which has presented some theoretical problems because of the large scattering length in this particular channel. These problems, having to do with the removal of divergences, have now been solved by the same authors[11] and almost simultaneously by Gegelia[12].

We will here consider a system of proton interacting with an antiproton. The experimental and theoretical situation has been reviewed by Batty[9] and also related effects in other hadronic atoms[10]. At the low energies we are interested, these particles can be considered to be without internal structure and thus be described by local Schrödinger fields ψ_1 and ψ_2. The free Lagrangian of one particles, with common mass m_p, is then

$$\mathcal{L}_0(\psi) = \psi^\dagger \left(i\frac{\partial}{\partial t} + \frac{1}{2m_p}\boldsymbol{\nabla}^2 \right) \psi \tag{1}$$

In the following we will ignore the spin degrees of freedom since our results apply to each value of the total spin quantum number of the combined system. It is described by a Lagrangian which can be split up into three main parts,

$$\mathcal{L} = \mathcal{L}_0(\psi_1) + \mathcal{L}_0(\psi_2) + \mathcal{L}_{EM} + \mathcal{L}_{had} \tag{2}$$

Here $\mathcal{L}_{EM}$ includes the electromagnetic interactions[3] between the particles and $\mathcal{L}_{had}$ gives their hadronic interactions. When the momenta of the particles are much smaller than the pion mass, their interactions due to both pion and vector meson exchanges can be described by local operators[6]. Assuming in the following that the system has total angular momentum $L = 0$, the two leading contact interactions can then be written as

$$\mathcal{L}_{had} = -C_0(\psi_1^\dagger\psi_1)(\psi_2^\dagger\psi_2) - C_2 \left[(\psi_1^\dagger\boldsymbol{\nabla}\psi_1) \cdot (\psi_2^\dagger\boldsymbol{\nabla}\psi_2) + h.c. \right] \tag{3}$$

where the effective coupling C_2 should be of the order C_0/M^2. The second term is therefore effectively reduced by p^2/M^2 compared with the first. It will be convenient to express these interactions in terms of the two-nucleon potential

$$\begin{aligned} V(\mathbf{p},\mathbf{q}) &= \langle \mathbf{p}|V|\mathbf{q}\rangle = C_0 + C_2(\mathbf{p}^2 + \mathbf{q}^2) \\ &\equiv V_0(\mathbf{p},\mathbf{q}) + V_2(\mathbf{p},\mathbf{q}) \end{aligned} \tag{4}$$

where $\mathbf{q}$ is the nucleon momentum in the initial state and $\mathbf{p}$ the momentum in the final state.

Let us now consider elastic scattering of these two hadrons due to the hadronic interaction. In the CM system the total energy of the two particles is then $E = p^2/2m$ where m is the reduced mass. We will first only take into account the contact term proportional with C_0 in (3). According to the counting rules established by Weinberg, the scattering
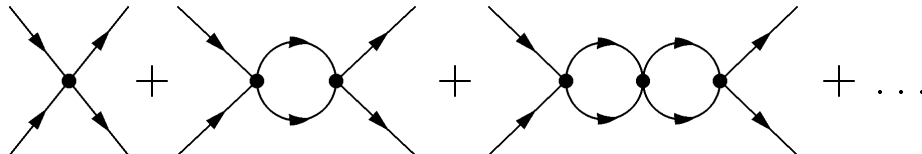

Figure 1: Chains of bubble diagrams gives the full scattering amplitude.

amplitude is given by the infinite sum of chains of bubbles as shown in Figure 1. The sum forms a geometric series and gives the T-matrix element[8]

$$T_0(p) = \frac{C_0}{1 - C_0 I(p)} \tag{5}$$

Here $I(E)$ is the integral of the free two-particle propagator,

$$I(p) = \int \frac{d^3k}{(2\pi)^3} \frac{2m}{\mathbf{p}^2 - \mathbf{k}^2 + i\epsilon} \tag{6}$$

It is seen to be linearly divergent in the ultraviolet. In order to regularize this and other similar power divergences, we will use minimal subtraction at a non-zero momentum $\mathbf{p}^2 = -\mu^2$ as suggested by Gegelia[12]. This is equivalent to PDS or power-divergence subtraction as proposed almost simultaneously by Kaplan, Savage and Wise[11]. The result is

$$I(p) = -\frac{m}{2\pi}(\mu + ip) \tag{7}$$

When used in (5), the scattering will then depend on the *renormalized* coupling constant C_0^R in this particular regularization scheme. It can be expressed in terms of the corresponding S-wave phase shift using the relations

$$p \cot \delta = ip - \frac{2\pi}{m} \frac{1}{T} \tag{8}$$

$$= -\frac{1}{a} + \frac{1}{2} r_0 p^2 + \dots \tag{9}$$

when introducing the scattering length a and effective range r_0. One then finds

$$\frac{1}{C_0^R(\mu)} = \frac{m}{2\pi} \left(\frac{1}{a} - \mu \right) \tag{10}$$

The effective range is zero in this lowest approximation.

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Now consider these two hadrons of opposite electrical charges bound by the Coulomb force into a hadronic atom. The unperturbed $1S$ ground has the energy $E_0 = -\gamma^2/2m$ where $\gamma = m\alpha$ is the momentum in the ground state with wave function $\Psi_0(r)$. It will be perturbed both by electromagnetic and hadronic interactions. The dominant part of the energy shift ΔE_0 will be due to the contact potential V_0 in (4). In lowest order perturbation theory the first contribution obviously is

$$\Delta E_0^{(1)}(C_0) = \langle \Psi_0 | V_0 | \Psi_0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \Psi_0^*(\mathbf{p}) V_0(\mathbf{p}, \mathbf{q}) \Psi_0(\mathbf{q}) = C_0 |\Psi_0(r = 0)|^2 \quad (11)$$

where the Fourier transform of the ground state wave function is

$$\Psi_0(\mathbf{p}) = \frac{8\pi^{1/2} \gamma^{5/2}}{(\mathbf{p}^2 + \gamma^2)^2} \quad (12)$$

Higher order contributions come from the same set of Feynman diagrams as in Figure 1. Their contributions again form a geometric series which gives

$$\Delta E_0(C_0) = |\Psi_0(0)|^2 \frac{C_0}{1 - C_0 I_\gamma} \quad (13)$$

where now

$$I_\gamma = \int \frac{d^3 k}{(2\pi)^3} \frac{-2m}{\mathbf{k}^2 + \gamma^2} \quad (14)$$

is the integral over the bound two-particle propagator. It is again linearly divergent. But after the same minimal subtraction as above, it becomes $I_\gamma^R = -(m/2\pi)(\mu - \gamma)$. Using this result together with $|\Psi_0(0)|^2 = \gamma^3/\pi$ we have

$$\Delta E_0(C_0) = \frac{\gamma^3}{\pi} \frac{C_0^R}{1 - C_0^R I_\gamma^R} = \frac{\gamma^3}{m} \frac{2a}{1 - a\gamma} \quad (15)$$

We see that the arbitrary renormalization point μ is dropping out of the result valid to all orders in the coupling constant as it should after renormalization. Introducing the Bohr radius $b = 1/\gamma$ which gives the size of the atom, the result can be written as

$$\Delta E_0(C_0) = m\alpha^2 \frac{2a}{b - a} = 2am^2 \alpha^3 \left[1 + \frac{a}{b} + \ldots \right] \quad (16)$$

since the scattering length $a \ll b$.

The leading term in this energy level shift was first derived by Deser *et al.* [13]. Since then corrections due to the electromagnetic interactions have been obtained by Trueman[14] and Kudryavtsev and Popov[15]. These effects can also be incorporated within the present framework and will be presented elsewhere[16].

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We will here consider instead the effect of the higher potential V_2 in (4). It gives a correction to the scattering amplitude which can be obtained by inserting this interaction at every vertex in the bubble diagrams in Figure 1. One then finds

$$\Delta T(p) = \frac{2C_2(C_0 I_0 + p^2)}{[1 - C_0 I(p)]^2} \tag{17}$$

after summing all the different contributions. The divergent integral

$$I_0 = -2m \int \frac{d^3 k}{(2\pi)^3} \tag{18}$$

is now zero in the regularization scheme we are using since it is independent of external momenta. On the other hand, $I(p)$ is the same integral as before and takes the finite value (7). Inverting the full scattering amplitude $T(p) = T_0(p) + \Delta T(p)$ and using Eqs.(8) and (9), we find that the renormalized coupling $C_0^R(\mu)$ is still given by (10) while $C_2^R(\mu)$ is related to the effective range,

$$C_2^R(\mu) = \frac{m r_0}{8\pi} C_0^R(\mu)^2 \tag{19}$$

as first obtained by Gegelia[12] and Kaplan, Savage and Wise[11].

Now let us consider the effect of V_2 on the bound state problem. The lowest order contribution to the energy level shift becomes

$$\begin{aligned}
\Delta E_0^{(1)}(C_0, C_2) &= \langle \Psi_0 | V_2 | \Psi_0 \rangle \\
&= C_2 (8\pi^{1/2} \gamma^{5/2})^2 \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{\mathbf{p}^2 + \mathbf{q}^2}{(\mathbf{p}^2 + \gamma^2)^2 (\mathbf{q}^2 + \gamma^2)^2} = 128\pi \gamma^5 C_2^R I_1 I_2^R
\end{aligned} \tag{20}$$

Here we have introduced the convergent integral

$$I_1 = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(\mathbf{p}^2 + \gamma^2)^2} = \frac{1}{8\pi\gamma} \tag{21}$$

and the divergent integral

$$I_2^R = \int \frac{d^3 p}{(2\pi)^3} \frac{\mathbf{p}^2}{(\mathbf{p}^2 + \gamma^2)^2} = \frac{1}{8\pi}(2\mu - 3\gamma) \tag{22}$$

which is made finite by the same regularization method as before. Higher order contributions from this interaction is now found by inserting it once in all the vertices in the bubble diagrams in Figure 1. After some resummations, we then find

$$\Delta E_0(C_0, C_2) = 2\frac{\gamma^4}{\pi} C_2^R \left[\frac{2(\mu - \gamma)}{1 - C_0^R I_\gamma^R} - \frac{\gamma}{(1 - C_0^R I_\gamma^R)^2} \right] \tag{23}$$

Because of the coupling constant renormalization condition (19), the last term is independent of the renormalization point μ. However, it appears explicitly in the first term so this

part of the result is not invariant under the corresponding renormalization group[11]. It is unusual to see the renormalization point appear linearly. In renormalizable theories it always appears logarithmically since the ultraviolet divergences there are logarithmic. The present effective theory, on the other hand, has power divergences in the ultraviolet which translate into explicit μ-dependence in the corresponding beta-functions for the running coupling constants[11]. We therefore have to accept also the explicit μ-dependence in our result.

However, as in more conventional renormalizable theories when we are working to a finite order in perturbation theory, we cannot expect our result, obtained here to first order in the coupling constant C_2, to be invariant under the renormalization group. Instead, we can exploit the would-be invariance to choose the arbitrary parameter μ in such a way as to maximally reduce its presence. Here, it can obviously be done by taking $\mu = \gamma$ which makes the first term go away. The resulting, total correction to the ground state energy is then

$$
\begin{aligned}
\Delta E_0 &= \frac{\gamma^3}{\pi} \left(\frac{C_0^R}{1 - C_0^R I_\gamma^R} - \frac{2\gamma^2 C_2^R}{[1 - C_0^R I_\gamma^R]^2} \right) \\
&= am^2\alpha^3 \left[\frac{2b}{b-a} - \frac{ar_0}{(b-a)^2} \right]
\end{aligned}
\tag{24}
$$

where the first part is from (16). It is possible also to obtain corrections to higher order in the effective range parameter. But we must then start out with an effective Lagrangian which involves higher derivative operators in the hadronic part (3).

The obtained results (16) and (24) applies both to the spin singlet and triplet component of the ground state since the total spin is conserved by the strong interactions. One can then obtain the hyperfine splitting in terms of the two corresponding scattering lengths and effective ranges. Protonium is obviously very unstable and will decay via hadronic annihilation. As a consequence, the energy levels will be smeared out because of these non-elastic channels. One can incorporate this effect in the present framework by letting the scattering lengths become complex[9]. More phenomenological applications of the obtained results to the protonium spectrum will be presented elsewhere together with higher order electromagnetic corrections[16].

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